Journal of Statistical Physics, Vol. 37, Nos. 3/4, 1984

Convergent Expansions for Asymptotically Degenerate Inverse Correlation Lengths of Classical Lattice Spin Systems

Michael O'Carroll[1,2]

Received December 22, 1983

We obtain convergent expansions for the inverse correlation length associated with various spin–spin correlation functions for some weakly coupled multicomponent classical lattice spin systems. In terms of the lattice quantum field theory associated with the models the expansions provide a convergent perturbation theory for particle masses which are asymptotically degenerate in the limit of zero coupling.

KEY WORDS: Multicomponent spin model; correlation length; correlation function; expansion for correlation length; analyticity of correlation length; high temperature expansions for degenerate correlation lengths.

1. INTRODUCTION

Recently in Refs. 1 and 2 convergent expansions were obtained for the inverse correlation length associated with the spin–spin (truncated spin–spin) correlation function (hereafter cf) of the high- (low-) temperature d-dimensional Ising model. Furthermore each term of the expansion can be computed by a finite algorithm. Similar methods and results hold for Ising-type models with more general single-spin distributions as well as for some gauge models.[3] From the point of view of the lattice quantum field theory associated with the model the expansion gives the mass of a particle. The mass is a point in the joint spectrum of the self-adjoint energy–momentum operators of the quantum field theory with zero momentum (see Ref. 4 for this connection). Formally $T = e^{-H}$ where T is the renormalized self-adjoint transfer matrix (sup spec $T = 1$) and H is the energy operator with inf spec

[1] Departamento de Física-ICEx, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil.

[2] Reseach partially supported by CNPq, Brazil.

0022-4715/84/1100-0439$03.50/0

$H=0$. From the point of view of spectral theory the expansion provides a convergent perturbation theory for a nondegenerate level.

In this paper we obtain convergent expansions for the masses of the quantum field theory associated with weakly coupled multicomponent classical spin systems. These masses are asymptotically degenerate in the limit of zero coupling. From the point of view of spectral theory we furnish a convergent perturbation theory for the case of levels which are asymptotically degenerate. The masses are related to certain components of spin–spin correlation functions in the limit of infinite separation of the spins. For the case of diagonal perturbations see Ref. 9.

For simplicity of exposition we consider the case of a doubly degenerate level and explicity the high-temperature classical rotor on the d-dimensional lattice Z^d in a magnetic field or with an anisotropic interaction or both. The Boltzmann factor is taken as

$$\exp\left\{\beta\left[\sum_{|x-y|=1}(S_x^1S_y^1+S_x^2S_y^2)+\varepsilon(S_x^1S_y^2+S_x^2S_y^1)\right]+h\sum_x S_x^1\right\}$$

where S_x^i is the ith component of the spin at the lattice site $x\in Z^d$ and β, ε, and h are taken to be small. The single-spin distribution is $\delta((S^1)^2+(S^2)^2-1)\,dS_1\,dS_2$. We denote points $x\in Z^d$ by $x=(x_1,\ \mathbf{x})$, $\mathbf{x}\in Z^{d-1}$ and $|x|\equiv\sum_{i=1}^d|x_i|=|x_1|+|\mathbf{x}|$. In this model the inverse correlation lengths

$$\xi_\alpha^{-1}\equiv\lim_{x_1\to\infty}\frac{-1}{x_1}\ln(\langle S_0^\alpha S_x^\alpha\rangle-\langle S_0^\alpha\rangle\langle S_x^\alpha\rangle),\qquad x=(x_1,\mathbf{0}),\ \alpha=1,2$$

are asymptotically ($\beta\downarrow 0$) degenerate and are degenerate for all β if $\varepsilon=h=0$. $\langle\,.\,\rangle$ denotes the thermodynamic limit average of . in the canonical ensemble with the above Boltzmann factor. The methods and results generalize to multicomponent spin models, gauge models where the character of the representation of the gauge group is complex[5] and to gauge-Higgs models.

Unlike the nondegenerate case where there is only one cf here we consider the 2×2 matrix-valued cf $G(x,\beta)$ with matrix elements given by

$$G_{\alpha\gamma}(x;\beta)=\lim_{\Lambda\to Z^d}G_{\Lambda\alpha\gamma}(y;z,\beta),\qquad x=y-z,\ \alpha,\gamma=1,2$$

where $G_{\Lambda\alpha\gamma}(y;z,\beta)\equiv\langle S_y^\alpha S_z^\gamma\rangle_\Lambda-\langle S_y^\alpha\rangle\langle S_z^\gamma\rangle$ and $\langle\,.\,\rangle_\Lambda$ denote averages for the finite lattice Λ. The existence, β analyticity, and translation invariance of $\lim_{\Lambda\uparrow Z^d}G_{\Lambda\alpha\gamma}(x,\beta)$ follow from the polymer expansion of Ref. 6.

In Section 2 we obtain a spectral representation for

$$\tilde{G}(p)=\frac{1}{(2\pi)^{d/2}}\sum_x G(x)\,e^{-ipx},\qquad p=(p_1,\mathbf{p}),\qquad p_x=\sum_{i=1}^d p_ix_i$$

the Fourier transform of $G(x)$, and develop the relevant spectral analysis to show that the mass points are contained in the set of zeros of det $\tilde{\Gamma}(p)$ for p_1 positive imaginary, $\mathbf{p}=0$, where $\tilde{\Gamma}(p)\,\tilde{G}(p)=1$, i.e., $\tilde{\Gamma}(p)$ is the Fourier transform of $\Gamma(x)$ where Γ is the 2×2 matrix convolution inverse of G, with matrix elements $\Gamma_{\alpha\rho}$. In other words

$$\sum_{\rho,z} G_{\alpha\rho}(x;z)\,\Gamma_{\rho\gamma}(z;y)=\delta_{xy}\delta_{\alpha\gamma}$$

In Section 3 we give convergent expansions for the mass points for the cases $\varepsilon\neq 0$, $h=0$; $\varepsilon=0$, $h\neq 0$; and $\varepsilon\neq 0$, $h\neq 0$. In Section 4 we explain how to generalize the methods to other degenerate situations and make some concluding remarks. In an appendix we establish regions of p_1 analyticity for $\tilde{G}(p)$ and $\tilde{\Gamma}(p)$ which are crucial for obtaining the expansions in Section 3.

2. SPECTRAL CONSIDERATIONS

Here we obtain a spectral representation for $\tilde{G}(p)$ analogous to the Kallen–Lehman representation in continuum quantum field theory and give criteria for a point to belong to the mass spectrum.

The lattice quantum field theory Hilbert space, with inner product denoted by (,), the energy-momentum and field operators are constructed as in Ref. 4. We denote by

$$E(\lambda_1) \text{ and } F(\boldsymbol{\lambda})=\prod_{i=2}^{d} F_i(\lambda_i), \qquad \boldsymbol{\lambda}\in(-\pi,\pi]^{d-1}$$

the spectral resolution of the self-adjoint evolution operator (renormalized transfer "matrix") momentum operator (generator of the unitary space translation operators), respectively. As in Ref. 4 we have the Feynman–Kac formula

$$G_{\alpha\gamma}(x,\beta)=\int_{[0,1)}\int_{(-\pi,\pi]^{d-1}} \lambda_1^{|x_1|}e^{i\boldsymbol{\lambda}\cdot\mathbf{x}}\,d\mu_{\alpha\gamma}(\lambda_1,\boldsymbol{\lambda})$$

where

$$\mu_{\alpha\gamma}(\lambda_1,\boldsymbol{\lambda})=(\hat{S}_0^{\alpha}\Omega, E(\lambda_1)\,F(\boldsymbol{\lambda})\,\hat{S}_0^{\gamma}\Omega)-(\hat{S}_0^{\alpha}\Omega,\Omega)(\Omega,\hat{S}_0^{\gamma}\Omega)$$

and $\hat{S}_0^{\alpha}\Omega(\Omega)$ is the Hilbert space vector associated with the function $S_0^{\alpha}(1)$. $[0,1)$ can be replaced by $[0,e^{-\ln c\beta})$ using the falloff of $G_{\alpha\gamma}(x=x_1,\mathbf{0}))$ given by Lemma A1. Adapting the proof of the results of Ref. 4 we have

Lemma 2.1. For each $\beta > 0$ and $\mathbf{p} \in (-\pi, \pi]^{d-1}$ there exist signed finite measures $d\rho_{\alpha\gamma}(\lambda_1, \mathbf{p})$, positive for $\alpha = \gamma$, such that

$$\tilde{G}_{\alpha\gamma}(p) = \int_{[0, e^{-\omega_{\alpha\gamma}(\mathbf{p})}]} \left(\frac{1 - \lambda_1^2}{1 - 2\lambda_1 \cos p_1 + \lambda_1^2} \right) d\rho_{\alpha\gamma}(\lambda_1, \mathbf{p})$$

where

$$\omega_{\sigma\sigma}(\mathbf{p}) \equiv \lim_{x_1 \to \infty} \frac{-1}{x_1} \ln \left(\sum_{\mathbf{x}} G_{\sigma\sigma}(x_1, \mathbf{x}))\, e^{i\mathbf{p}\cdot\mathbf{x}} \right)$$
$$\geqslant \omega_{\sigma\sigma}(\mathbf{0}) \equiv m_{\sigma\sigma} > -\ln c\beta$$

$\omega_{\alpha\gamma}(\mathbf{p}) = \min\{\omega_{\alpha\alpha}, \omega_{\gamma\gamma}\}$ for $\alpha \neq \gamma$. Furthermore for a product of intervals $\Delta_1 \times \mathbf{\Delta}$, $\int_{\Delta_1} d\rho_{\alpha\alpha}(\lambda_1, \mathbf{p})$ is continuous in $\mathbf{p} \in (-\pi, \pi]^{d-1}$ and

$$\mu_{\alpha\alpha}(\Delta_1 \times \mathbf{\Delta}) = \int_{\mathbf{\Delta}} \left[\int_{\Delta_1} d\rho_{\alpha\alpha}(\lambda_1, \mathbf{p}) \right] d\mathbf{p}$$

Remarks. 1. The importance of the above formula is that it relates the energy momentum spectrum, i.e., the suport of the $d\mu_{\alpha\alpha}$, to the support of the measures $d\rho_{\alpha\alpha}(\lambda_1, \mathbf{p})$.

2. Formally $d\rho_{\alpha\gamma}(\lambda_1, \mathbf{p}) = \int_{(-\pi,\pi]^{d-1}} \delta(\boldsymbol{\lambda} - \mathbf{p})\, d\mu_{\alpha\gamma}(\lambda_1, \boldsymbol{\lambda})$.

From the above lemma we see that to locate the mass spectrum it is enough to determine the support of the measures $d\rho_{\alpha\alpha}(\lambda_1, \mathbf{p} = 0)$. From now on we suppress the $\mathbf{p}$ dependence and take $\mathbf{p} = 0$. We now express $\tilde{G}_{\alpha\gamma}(p)$ in a more convenient "resolvent" form by introducing the spectral parameter a and measures $d\nu_{\alpha\gamma}(a)$ defined by

$$d\nu_{\alpha\gamma}(a) = \frac{1 - \lambda(a)^2}{2\lambda(a)}\, d\rho_{\alpha\gamma}(\lambda(a)), \qquad a(\lambda) = (1 - \lambda)^2/2\lambda$$

and $a(e^{-m_{\alpha\beta}}) = \cosh m_{\alpha\beta} - 1$, so that $\tilde{G}(p) = F(z = \cos p_1 - 1)$ where we define

$$F_{\alpha\gamma}(z) = \rho_{\alpha\gamma}(\{\lambda = 0\}) + \int_{\cosh m_{\alpha\gamma} - 1}^{\infty} \frac{d\nu_{\alpha\gamma}(a)}{a - z}$$

$F_{\alpha\gamma}(z)$ is analytic in $z \in C - [\cosh m_{\alpha\gamma} - 1, \infty)$ and we have set $m_{\alpha\gamma} = \omega_{\alpha\gamma}(\mathbf{0})$.

We recall the well-known inversion foruma in the following:

Lemma 2.2. If c, $d (c < d)$ are points of continuity of $d\nu_{\alpha\alpha}$ then

$$\nu_{\alpha\alpha}(d) - \nu_{\alpha\alpha}(c) = \lim_{\varepsilon \downarrow 0} \frac{1}{i\pi} \int_c^d [F_{\alpha\alpha}(v + i\varepsilon) - F_{\alpha\alpha}(v - i\varepsilon)]\, dv$$

Remark. The "resolvent" representation of $F(z)$ and the inversion formula can be used in the spectral analysis of Refs. 1–4, making the representation theorem for Herglotz functions unnecessary. In the nonmatrix case $\operatorname{Im} F(z)^{-1} < 0$ for $\operatorname{Im} z > 0$ since $\operatorname{Im} F(z) > 0$ for $\operatorname{Im} z > 0$ from the representation for $F(z)$. Furthermore for $\operatorname{Im} z = 0$, $\operatorname{Re} z$ sufficiently negative, $\operatorname{Im} F(z) = 0$ so that $\operatorname{Im} F(z)^{-1} = 0$ and using the Cauchy–Riemann equations we conclude F^{-1} is monotone. Thus in the F^{-1} analyticity region F^{-1} has at most one zero. F^{-1} analyticity is used in the inversion formula by setting

$$F(v + i\varepsilon) - F(v - i\varepsilon) = 1/F(v + i\varepsilon)^{-1} - 1/F(v - i\varepsilon)^{-1}$$

Let $H(z)$ be the matrix inverse of $F(z)$, i.e., $HF = 1$, and $F_{\alpha\alpha}(z) = H_{\gamma\gamma}(z)/\det H(z)$, $\alpha \neq \gamma$. Note that $H(z = \cos p_1 - 1) = \Gamma(p_1) \equiv \tilde{\Gamma}(p_1,\ \mathbf{p} = 0)$ and that by Lemma A1 $\tilde{\Gamma}(p_1)$ and $\det \tilde{\Gamma}(p_1)$ are analytic in $0 < \operatorname{Im} p_1 < -2 \ln c'\beta$ so that by $F_{\alpha\alpha} = H_{\gamma}\gamma/\det H$ and Lemma 2.2 we conclude that the mass spectrum is discrete in $(0, -2 \ln c'\beta)$. We give criteria for a point $p_1 = im$ to, or not to, belong to the mass spectrum $\sigma(M)$.

Lemma 2.3. Let $0 < m < -2 \ln c'\beta$

(a) If $\det \tilde{\Gamma}(p_1 = im) \neq 0$ then $m \notin \sigma(M)$,

(b) If $\det \tilde{\Gamma}(p_1 = im) = 0$ and $\Gamma_{\alpha\alpha}(p_1 = im) \neq 0$, $\alpha = 1$ or 2 then $m \in \sigma(M)$.

Remark. It cannot happen that $\det \tilde{\Gamma}(p_1) = 0$ and $\tilde{\Gamma}_{\alpha\alpha}(p_1) \neq 0$, $\alpha = 1$ or 2, for $0 < \operatorname{Im} p_1 < -2 \ln c'\beta$, $|\operatorname{Re} p_1| < \pi$, but $\operatorname{Re} p_1 \neq 0$ since $\tilde{G}_{\alpha\alpha} = \tilde{\Gamma}_{\gamma\gamma}/\det \tilde{\Gamma}$, $\alpha \neq \gamma$, is analytic at these points.

In the following section we introduce and solve implicit equations for the zeros of $\det \tilde{\Gamma}(p_1)$.

3. IMPLICIT EQUATIONS AND CONVERGENT EXPANSIONS FOR THE MASSES

In this section we give convergent expansions for the masses for the cases $\varepsilon \neq 0$, $h = 0$; $\varepsilon = 0$, $h \neq 0$; and $\varepsilon \neq 0$, $h \neq 0$. The implicit equations for the masses can be solved by the methods of Refs. 1 and 5 so we only sketch the results. In obtaining the results below we use the falloff of G and Γ given by Lemma A1. Also we use the $\beta = 0$ expansions of $G(x, \beta)$ given in Lemma A4. The subscript S means the constant and linear term in the $\beta = 0$ Taylor expansion is to be subtracted.

3.1. $\epsilon \neq 0,\ h = 0$

We see that $G_{11}(x) = G_{22}(x)$ so that $\tilde{G}_{11} = \tilde{G}_{22}$ and $\tilde{\Gamma}_{11} = \tilde{\Gamma}_{22}$. Thus $\det \tilde{\Gamma} = (\tilde{\Gamma}_{11} + \tilde{\Gamma}_{12})\ (\tilde{\Gamma}_{11} - \tilde{\Gamma}_{12})$, i.e., $\det \tilde{\Gamma}$ factorizes and the masses are contained in the zeros of the functions $\tilde{R}^{\pm} \equiv \tilde{\Gamma}_{11} \pm \tilde{\Gamma}_{12}$ for p_1 positive imaginary. We carry out the $\beta = 0$ Taylor expansions for $\tilde{G}(p_1)$ and $\tilde{\Gamma}(p_1)$ as in Ref. 5 but using the result of Lemma A4 to obtain

Lemma 3.1:

(a) $\tilde{G}(p_1) =$

$$\left(\begin{array}{c|c} \frac{1}{2} + \frac{\beta}{2}(d-1) + \frac{\beta}{4}(e^{-ip_1} + e^{ip_1}) & \frac{\beta\varepsilon}{2}(d-1) + \frac{\beta\varepsilon}{4}(e^{-ip_1} + e^{ip_1}) \\ \hline \frac{\beta\varepsilon}{2}(d-1) + \frac{\beta\varepsilon}{4}(e^{ip_1} + e^{ip_1}) & \frac{1}{2} + \frac{\beta\varepsilon}{2}(d-1) + \frac{\beta}{4}(e^{-ip_1} + e^{ip_1}) \end{array}\right) + \tilde{G}_S(p_1)$$

(b) $\tilde{\Gamma}(p_1) =$

$$\left(\begin{array}{c|c} \frac{1}{2} - \frac{\beta}{4}(e^{-ip_1} + e^{ip_1}) - \frac{\beta}{2}(d-1) & -\frac{\beta}{4}(e^{-ip_1} + e^{ip_1}) - \frac{\beta\varepsilon}{2}(d-1) \\ \hline -\frac{\beta\varepsilon}{4}(e^{-ip_1} + e^{ip_1}) - \frac{\beta\varepsilon}{2}(d-1) & \frac{1}{2} - \frac{\beta}{4}(e^{-ip_1} + e^{ip_1}) - \frac{\beta}{2}(d-1) \end{array}\right) + \tilde{\Gamma}_S(p_1)$$

From Lemma 3.1 we can write

$$\tilde{R}^{\pm}(p_1, \beta) = \frac{1}{2} - \frac{\beta}{4}(1 \pm \varepsilon)(e^{-ip_1} + e^{ip_1}) - \frac{\beta}{2}(d-1)(1 \pm \varepsilon) + R_S^{\pm}(p_1, \beta)$$

The implicit equations for the masses are solved by introducing the auxiliary complex variables $w_{\pm}$ and functions $H^{\pm}(w_{\pm}, \beta)$ such that $H^{\pm}(w_{\pm} = 1/2 - (\beta/4)(1 \pm \varepsilon)e^{-ip_1}, \beta) = \tilde{R}^{\pm}(p_1, \beta)$. We find that $H^{\pm}(w, \beta)$ is jointly analytic, $H^{\pm}(0, 0) = 0$ and $(\partial H^{\pm}/\partial w)(0, 0) = 1$. Thus by the analytic implicit function theorem there exist $w_{\pm}(\beta)$ analytic at $\beta = 0$, $w_{\pm}(0) = 0$, such that $H^{\pm}(w_{\pm}(\beta), \beta) = 0$. Thus there are two masses given by

$$m_{\pm}(\beta) = -\ln \beta + \ln 2 - \ln(1 \pm \varepsilon) + \ln(1 - 2w_{\pm}(\beta))$$

and there is mass splitting in the β-independent term.

3.2. $\epsilon = 0,\ h \neq 0$

We see that $\tilde{G}(p)$ and $\tilde{\Gamma}(p)$ are diagonal so that $\det \tilde{\Gamma}(p_1) = \tilde{\Gamma}_{11}(p_1)\tilde{\Gamma}_{22}(p_1)$ and the masses are contained in the zeros of $\tilde{\Gamma}_{11}(p_1)$ and

$\tilde{\Gamma}_{22}(p_1)$ for p_1 positive imaginary. In terms of the J functions of the appendix and using Lemma A4 we have the following $\beta=0$ Taylor expansions:

Lemma 3.2.:

(a) $\tilde{G}_{11}=J_0-J_1^2+\beta[2d(J_3J_1-3J_0J_1^2-2J_1^4)+2(d-1+\cos p_1)(J_0-J_1^2)^2]$
$\quad + \tilde{G}_{S11}$

(b) $\tilde{G}_{22}=J_4+\beta(J_5J_1-2dJ_4J_1^2+2(d-1+\cos p_1)J_4^2)$
$\quad + \tilde{G}_{S22}$

(c) $\tilde{\Gamma}_{11}=(J_0-J_1^2)^{-1}-\beta(J_0-J_1^2)^{-2}[2d(J_3J_1-3J_0J_1^2-2J_1^4)$
$\quad +2(d-1)(J_0-J_1^2)^2]-\beta(e^{-ip_1}+e^{ip_1})+\tilde{\Gamma}_{S11}$

(d) $\tilde{\Gamma}_{22}=J_4^{-1}-\beta J_4^{-2}(J_5J_1-2dJ_4J_1^2+2(d-1)J_4^2)$
$\quad -\beta(e^{-ip_1}+e^{ip_1})+\tilde{\Gamma}_{S22}$

To solve $\tilde{\Gamma}_{11}(p_1,\beta)=0$ introduce w and $H_1(w,\beta)$ such that

$$H_1(w=(J_0-J_1^2)^{-1}-\beta e^{-ip_1},\beta)=\tilde{\Gamma}_{11}(p_1,\beta)$$

We find that $\tilde{\Gamma}_{11}(p_1=im_1,\beta)=0$ where

$$m_1=-\ln\beta+\ln[(J_0-J_1^2)^{-1}-w_1(\beta,h)]$$

and $H_1(w_1(\beta),\beta)=0$, $w_1(0)=0$ and $w_1(\beta)$ is analytic at $\beta=0$.

To solve $\tilde{\Gamma}_{22}(p_1,\beta)=0$ introduce w and $H_2(w,\beta)$ such that

$$H_2(w=J_4^{-1}-\beta e^{-ip_1},\beta)=\tilde{\Gamma}_{22}(p_1,\beta)$$

We find that $\tilde{\Gamma}_{22}(p_1=im_2,\beta)=0$ where

$$m_2=-\ln\beta+\ln(J_4^{-1}-w_2(\beta))$$

and $H_2(w_2(\beta),\beta)=0$, $w_2(0)=0$, and $w_2(\beta)$ is analytic at $\beta=0$.

Using Lemma A2 we find

$$m_1=-\ln\beta-\ln\frac{1}{2}+\frac{3}{8}h^2+O(h^4)+O(\beta)$$

and

$$m_2=-\ln\beta-\ln\frac{1}{2}+\frac{h^2}{8}+O(h^4)+O(\beta)$$

so that there is mass splitting.

3.3. $\epsilon \neq 0,\ h \neq 0$

There is no apparent factorization of det $\tilde{\Gamma}(p_1)$. The $\beta = 0$ expansions of $\tilde{G}(p_1)$ and $\tilde{\Gamma}(p_1)$ are given by the following.

Lemma 3.3:

$$\text{(a)}\ \tilde{G}(p_1) = \left(\begin{array}{c|c} a_{11} + b_{11}\beta & b_{12}\beta \\ \hline b_{12}\beta & a_{22} + b_{22}\beta \end{array}\right) + \tilde{G}_S(p_1)$$

$$\text{(b)}\ \tilde{\Gamma}(p_1) = \left(\begin{array}{c|c} \dfrac{1}{a_{11}} - \dfrac{b_{11}}{a_{11}^2} & -\dfrac{b_{12}}{a_{11}a_{22}}\beta \\ \hline -\dfrac{b_{12}}{a_{11}a_{22}} & \dfrac{1}{a_{22}} - \dfrac{b_{22}}{a_{22}^2}\beta \end{array}\right) + \tilde{\Gamma}_S(p_1)$$

where

$$a_{11} = J_0 - J_2^2, \qquad b_{11} = 2d(J_3J_1 - 3J_0J_1^2 - 2J_1^4) + 2(d-1)(J_0 - J_1^2)^2 + 2\cos p_1(J_0 - J_1^2)^2$$

$$a_{22} = 1 - J_0, \qquad b_{22} = (J_5J_1 - 2dJ_4J_1^2) + 2(d-1)(1-J_0)^2 + 2\cos p_1(1-J_0)^2$$

$$b_{12} = \varepsilon(1-J_0)(J_0 - J_1^2) + \varepsilon(1-J_0)(J_0 - J_1^2)(2d-1) + 2\cos p_1\varepsilon(1-J_0)(J_0 - J_1^2)$$

It is convenient to write $\tilde{\Gamma}(p_1)$ and det $\tilde{\Gamma}(p_1)$ as in the following.

Lemma 3.4:

$$\text{(a)}\ \tilde{\Gamma} = \left(\begin{array}{c|c} a_{11}^{-1} + \gamma_{11}\beta - 2\cos p_1\beta & -\beta\varepsilon 2\cos p_1 + \gamma_{12}\beta \\ \hline -\beta\varepsilon 2\cos p_1 + \gamma_{12}\beta & a_{22}^{-1} + \gamma_{22}\beta - 2\cos p_1\beta \end{array}\right) + \tilde{\Gamma}_S$$

$$\begin{aligned}\text{(b)}\ \det\tilde{\Gamma}(p_1) = {} & [\underline{a_{11}^{-1}} + \gamma_{11}\beta - \underline{\beta(e^{-ip_1}} + e^{ip_1}) + \tilde{\Gamma}_{S_{11}}] \\ & \times [\underline{a_{22}^{-1}} + \gamma_{22}\beta - \underline{\beta(e^{-ip_1}} + e^{ip_1}) + \tilde{\Gamma}_{22}] \\ & - [\underline{-\beta\varepsilon(e^{-ip_1}} + e^{ip_1}) + \gamma_{12}\beta + \tilde{\Gamma}_{S12}]^2\end{aligned}$$

where

$$\gamma_{11} = -(J_0 - J_1^2)^{-1}[2d(J_3J_1 - 3J_0J_1^2 - 2J_1^4) + 2(d-1)(J_0 - J_1^2)^2]$$

$$\gamma_{22} = -(1-J_0)^2[(J_5J_1 - 2dJ_4J_1^2) + 2(d-1)(1-J_0)^2]$$

$$\gamma_{12} = -\varepsilon 2d$$

To solve $\det \tilde{\Gamma} = 0$ introduce w and functions $H_{\alpha\gamma}(w, \beta)$ such that $H_{\alpha\gamma}(w = a_{11}^{-1} - \beta e^{-ip_1}, \beta) = \tilde{\Gamma}_{\alpha\gamma}(p_1, \beta)$. Thus we can write

$$\begin{aligned} \det H(w, \beta) &= (1 - \varepsilon^2) w^2 + (-a_{11}^{-1} + a_{22}^{-1} + 2\varepsilon^2 a_{11}^{-1}) w - \varepsilon^2 a_{11}^{-1} + T(w, \beta) \\ &\equiv P(w, \beta) + T(w, \beta) \end{aligned}$$

where $P(w, \beta)$ takes into accout the underlined terms in Lemma 3.4b, $\det H(w, \beta)$ is jointly analytic and $T(w, \beta) = \sum_{k \geqslant 1} t_{jk} w^j \beta^k$. $P(w, \beta)$ can be written $P(w, \beta) = a(w - c_0^-)(w - c_0^+)$ where

$$\begin{aligned} a &= (1 - \varepsilon^2) \\ c_0^{\pm} &= \{-[-a_{11}^{-1} + a_{22}^{-1} + 2\varepsilon^2 a_{11}^{-1}] \pm [(-a_{11}^{-1} + a_{22}^{-1} + 2\varepsilon^2 a_{11}^{-1})^2 \\ &\quad + 4(1 - \varepsilon^2)\, \varepsilon^2 (a_{11}^{-2})]^{1/2}\}/2(1 - \varepsilon^2) \end{aligned}$$

Now $\det H(w = c_0^{\pm}, 0) = 0$ and $(\partial H/\partial w)(w = c_0^{\pm}, 0) = \pm(c_0^+ - c_0^-) a \neq 0$. Thus there are two zeros of $\det H$ given by $w_{\pm} = c_0^{\pm} + u_{\pm}$, $u_{\pm}(\beta) = O(\beta)$ so that $w_{\pm} = a_{11}^{-1} - \beta e^{m_{\pm}}$ or the two masses are

$$m_{\pm} = -\ln \beta + \ln(a_{11}^{-1} - c_0^{\pm}) + \ln[1 + (a_{11}^{-1} - c_0^{\pm}) u_{\pm}]$$

Considering the small-h behavior of $a_{11}^{-1} - c_0^{\pm}$ we have

$$\begin{aligned} m_{\pm} \simeq & -\ln \beta + \ln 2 + \frac{3}{8} h^2 - \left\{ \frac{(2^{-1} h^2 - 4\varepsilon^2)}{4(1 - \varepsilon^2)} \right. \\ & \left. \pm \frac{[(4\varepsilon^2 - h^2 2^{-1})^2 + 16(1 - \varepsilon^2)\varepsilon^2]^{1/2}}{4(1 - \varepsilon^2)} \right\} + O(\beta) \end{aligned}$$

4. GENERALIZATIONS AND CONCLUDING REMARKS

In order to find inverse correlation lengths or masses for $O(n)$ spin systems and their perturbations an appropriate spin–spin $n \times n$ matrix $\mathcal{G}$ can be introduced. A spectral representation can be obtained as in Section 2. Again the masses will be contained in the set of positive, imaginary p_1 such that $\det \tilde{\Gamma}(p_1) = 0$ for the appropriate $\tilde{\Gamma}(p_1)$. Depending on the form of the perturbation $\det \tilde{\Gamma}(p_1)$ may or may not factorize. If it factorizes one determines the zeros of each factor. In any case by the introduction of an auxiliary complex variable w and jointly analytic function $\det H(w, \beta)$ the number of zeros or masses of $\det H(w, \beta)$ may be analyzed using the Weierstrass preparation theorem.[7] The techniques of Ref. 8 for the analysis of algebraic curves can be used to obtain explicit expansions for the masses.

It would be interesting to study the analyticity of the masses in the other parameters of the perturbations. For example the mass of the Ising model in a weak magnetic field of strength h is given by $m(\beta, h) = -\ln \beta + r(\beta, h)$ where $r(\beta, h)$ is jointly analytic in β and h. Also the problem of a convergent perturbation theory for the masses of the time-continuum version of these lattice models remains open.

APPENDIX

Using the decoupling of the hyperplane method as in Ref. 1 we obtain decay of G and Γ for small $|\beta|$ as the following.

Lemma A1:

(a) $|G_{\alpha\gamma}(x, \beta)| \leqslant c_1 \, |c\beta|^{|x_1| + |\mathbf{x}|}$

(b) $|\Gamma_{\alpha\gamma}(x, \beta)| \leqslant c_2 \, |c'\beta|^{2|x_1| + |\mathbf{x}|}$

except for $x = (\pm 1, \mathbf{0})$, $\alpha = \gamma$: for $x = (\pm 1, \mathbf{0})$, $\alpha = \gamma$ replace the 2 by 1.

In the $\beta = 0$ expansion of $G(x, \beta)$, $\tilde{G}(\beta)$ and $\tilde{\Gamma}(p)$ we use the h-dependent constants $J_0(h)$, $J_1(h)$, and $J_3(h)$ defined by (suppressing the h dependence)

$$(J_0, J_1, J_3) = \int (\cos^2 \phi, \cos \phi, \cos^3 \phi) \exp(h \cos \phi) \, d\phi$$

$$\times \left[\int \exp(h \cos \phi) \, d\phi \right]^{-1}$$

We also set $J_4 = 1 - J_0$ and $J_5 = J_1 - J_3$. Their small-h behavior is given by the following.

Lemma A2. J_0, J_1, J_3 are analytic for small $|h|$ and

(a) $J_0 = \dfrac{1}{2} + \dfrac{1}{16} h^2 + O(h^4)$

(b) $J_1 = \dfrac{h}{2} + O(h^3)$

(c) $J_3 = O(h)$

By expanding the numerator and denominator of $\langle \cdot \rangle_\Lambda$ and passing to the $\Lambda \uparrow Z^d$ limit we obtain the $\beta = 0$ expansions for the cf needed in Section 3. We have

Lemma A3:

(a) $\langle S_0^1 \rangle = J_1 + 2d\beta J_1(J_0 - J_1^2) + O(\beta^2)$

(b) $\langle S_0^2 \rangle = \beta\varepsilon J_1(1 - J_0)2d + O(\beta^2)$

(c) $\langle (S_0^1)^2 \rangle = J_0 + 2d\beta[J_3 J_1 - J_0 J_1^2] + O(\beta^2)$

(d) $\langle (S_0^2)^2 \rangle = J_4 + \beta(J_5 J_1 - 2dJ_4 J_1^2) + O(\beta^2)$

and for $|x| = 1$,

(e) $\langle S_0^1 S_x^1 \rangle = \beta(J_0 - J_1^2)^2 + O(\beta^2)$

(f) $\langle S_0^2 S_x^2 \rangle = \beta J_4^2 + O(\beta^2)$

(g) $\langle S_0^1 S_x^2 \rangle = \beta\varepsilon[J_0(1 - J_0) + (2d - 1) J_1^2(1 - J_0)] + O(\beta^2)$

The $\beta = 0$ expansions of $G_{\alpha\gamma}(x, \beta)$ for $x = 0$ and $|x| = 1$ are obtained from Lemma A3 as

Lemma A4. For $x = 0$,

(a) $G_{11} = J_0^2 - J_1^2 + 2d\beta[J_3 J_1 - 3J_0 J_1^2 - 2J_1^4] + O(\beta)^2$

(b) $G_{22} = J_4 + \beta(J_5 J_1 - 2dJ_4 J_1^2) + O(\beta^2)$

(c) $G_{12} = 2\beta\varepsilon\, d(J_1^2 - J_1 J_3) - \beta\varepsilon\, 2d(1 - J_0)J_1^2 + O(\beta^2)$

and for $|x| = 1$,

(d) $G_{11} = \beta(J_0 - J_1^2)^2 + O(\beta^2)$

(e) $G_{22} = \beta(1 - J_0)^2 + O(\beta^2)$

(f) $G_{12} = \beta\varepsilon(1 - J_0)(J_0 - J_1^2) + O(\beta^2)$

REFERENCES

1. M. O'Carroll, *J. Stat. Phys.* **34**:597–608 (1984); *Phys. Lett. B* **143**:188–192 (1984).
2. M. O'Carroll and W. Barbosa, *J. Stat. Phys.* **34**:609–614 (1984).
3. M. O'Carroll and G. Braga, *J. Math. Phys.* **25**:2741–2743 (1984).
4. R. Schor, *Commun. Math. Phys.* **59**:213 (1978).
5. M. O'Carroll, G. Braga, and R. Schor, (to appear *Commun. Math. Phys.* 1984).
6. E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics* (Lecture Notes in Physics No. 159, Springer, New York, 1982).
7. A. I. Markusevich, *Theory of Functions of a Complex Variable*, Vol. II (Prentice-Hall, Englewood Cliffs, New Jersey, 1965).
8. E. Hille, *Analytic Functions*, Vol. II (Ginn and Company, Boston, 1962).
9. M. Fisher and D. Ritchie, *Phys. Rev. B* **5**:2668 (1972).